

Complete and Deterministic Bell State Measurement Using Nonlocal Spin Products

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A simple protocol for complete and deterministic Bell state measurement is proposed. It consists of measurements of nonlocal spin product operators with the help of shared entanglement as an ancillary resource. The protocol realizes not only nonlocal Bell state measurement between a pair of distant qubits but also a complete Bell filter that transmits either one of the Bell states indicated by the measurement outcome. These schemes will be useful in quantum technologies where nonlocal Bell state measurement is indispensable.

INTRODUCTION

Bell state measurement, or Bell measurement, is an essential concept in quantum technologies [1]. The most simple Bell measurement scheme is illustrated in Fig. 1. Although this scheme is simple and instructive, it requires a nonlocal controlled-not (CNOT) operation between the two qubits. A deterministic CNOT operation and thus a complete Bell measurement would be possible in various qubit systems where two qubits are situated close to each other. However, Bell measurement between distant qubits is not possible using only local operation and classical communication (LOCC). More realistic implementations of Bell measurement using linear optics [2] are sometimes applied to various proof-of-principle demonstrations of quantum information protocols such as quantum teleportation and entanglement swapping etc. [3–6], but it is known that they cannot be complete and deterministic, i.e., the linear-optical implementations can only partly distinguish among the four possible Bell states [2, 7].

In this letter, a simple scheme for complete and deterministic Bell measurement is proposed. It consists of the measurements of *nonlocal spin product operators*, the members of nonlocal product observables [8], with the help of shared entanglement as a resource. The protocol enables nonlocal Bell measurement between a pair of distant qubits. Furthermore, not only Bell measurement but also a complete Bell filter is realizable, where the output state turns out to be either one of the Bell states indicated by the measurement outcome.

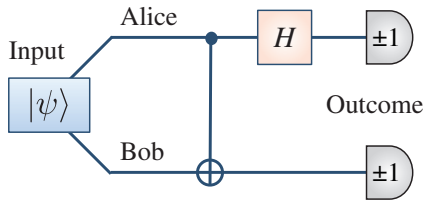


FIG. 1: A simple circuit model for the Bell measurement composed of a CNOT gate followed by a Hadamard gate [1].

NONLOCAL SPIN PRODUCTS AND BELL BASES

We consider a bipartite qubit system where a pair of qubits is distributed between Alice and Bob. In general, the state vector $|\Psi\rangle$ of the bipartite system is expressed as

$$|\psi\rangle = \sum_{\mu,\nu} c_{\mu\nu} |\mu\nu\rangle, \quad (1)$$

where $|\mu\nu\rangle \equiv |\mu\rangle_A \otimes |\nu\rangle_B$ and $|\mu\rangle_A$ ($|\nu\rangle_B$) is the eigenstate of the Pauli operator σ_z on Alice's (Bob's) site having eigenvalues μ , $\nu = \pm 1$. Note that $\sum_{\mu,\nu} |c_{\mu\nu}|^2 = 1$. Hereafter we sometimes write just + and – for the eigenvalues +1 and –1, respectively. For instance, $|+-\rangle = |+\rangle_A \otimes |-\rangle_B = | +1\rangle_A \otimes | -1\rangle_B$. Also note that we regard the states $|+\rangle$ and $|-\rangle$ as $|0\rangle$ and $|1\rangle$ in the standard qubit representation, respectively. The Bell bases are defined as

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}} (|++\rangle \pm |--\rangle), \quad (2)$$

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}} (|+-\rangle \pm |-+\rangle). \quad (3)$$

Using the Bell bases, $|\psi\rangle$ in (1) is rewritten as

$$|\psi\rangle = c_1 |\Phi^+\rangle + c_2 |\Phi^-\rangle + c_3 |\Psi^+\rangle + c_4 |\Psi^-\rangle, \quad (4)$$

where $c_1 = (c_{++} + c_{--})/\sqrt{2}$, etc.

The simple circuit for the Bell measurement given in Fig. 1 transforms the input states $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi^+\rangle$ and $|\Psi^-\rangle$ into the output states $|++\rangle$, $|+-\rangle$, $|-+\rangle$ and $|--\rangle$, respectively. Thus we can distinguish all the Bell states by observing the local measurement outcomes on Alice's and Bob's qubits. However, as noted earlier, the simple scheme requires nonlocal CNOT operation between distant qubits.

Nonlocal spin product operators, or nonlocal spin products, S_{ij} are expressed as

$$S_{ij} \equiv \sigma_i \otimes \sigma_j, \quad (i, j = x, y, z) \quad (5)$$

where the first and second Pauli operators act on Alice's and Bob's qubits, respectively. The eigenvalues of S_{ij} are

± 1 and each of them is doubly degenerate. For instance, eigenvectors of S_{zz} for eigenvalues $m = +1$ and -1 are written as

$$|m = +1\rangle = \alpha|\Phi^+\rangle + \beta|\Phi^-\rangle, \quad (6)$$

$$|m = -1\rangle = \gamma|\Psi^+\rangle + \delta|\Psi^-\rangle. \quad (7)$$

Importantly, we find

$$[S_{ii}, S_{jj}] = 0. \quad (8)$$

Thus, for instance, S_{zz} commutes with S_{xx} . As a result, S_{zz} and S_{xx} are compatible having a complete orthonormal set of common eigenbases:

$$|m = +1, n = \pm 1\rangle = |\Phi^\pm\rangle, \quad (9)$$

$$|m = -1, n = \pm 1\rangle = |\Psi^\pm\rangle, \quad (10)$$

where n refers to the eigenvalue of S_{xx} . These are nothing other than the Bell bases (2) and (3). Thus, by observing S_{zz} and S_{xx} for a given input state, we carry out a complete Bell measurement.

MEASUREMENT OF THE SPIN PRODUCTS

Suppose Alice and Bob want to measure any nonlocal spin product S_{ij} for an arbitrary system state $|\psi\rangle_S$ expressed in (1) or (4). Here, the suffix S after the state vector refers to the system to be measured.

Measuring a component of S_{ij} is simple. Since the measurements of σ_i by Alice and σ_j by Bob are compatible, we can make simultaneous local measurements of σ_i and σ_j , and then compute the product of Alice's and Bob's outcomes to obtain the measurement result of S_{ij} . Consider, for example, the measurement of S_{zz} . The measurement operators $M(\mu\nu)$ for the four possible combinations of Alice's and Bob's outcomes, (μ, ν) , are the projective operators:

$$M(\mu\nu) = \Pi(\mu\nu) = |\mu\nu\rangle\langle\mu\nu|, \quad (11)$$

where $\Pi(\dots)$ is the projector to the state $|\dots\rangle$. The corresponding POVM (positive operator valued measure) is $E(\mu\nu) = M^\dagger(\mu\nu)M(\mu\nu) = \Pi(\mu\nu)$. Taking a product of μ and ν , we obtain the outcome m for the measurement of S_{zz} . The POVMs E_\pm for $m = \pm 1$ are obtained as

$$E_+ = \Pi(++) + \Pi(--) = \Pi(\Phi^+) + \Pi(\Phi^-), \quad (12)$$

$$E_- = \Pi(+-) + \Pi(-+) = \Pi(\Psi^+) + \Pi(\Psi^-). \quad (13)$$

We see that E_\pm correspond to the projection to the eigenspaces of $m = \pm 1$ presented in (6) and (7), respectively.

However, this local measurement strategy is inappropriate to make simultaneous measurements of two or more components of S_{ij} , for instance, S_{zz} and S_{xx} . Since

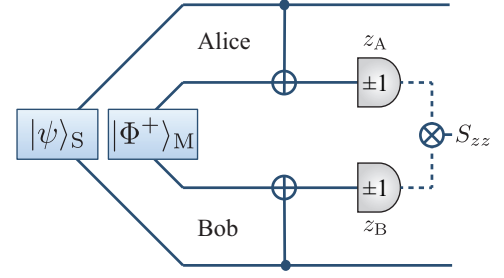


FIG. 2: A scheme for the measurement of nonlocal spin product S_{zz} .

the above-mentioned measurement of S_{zz} projects the system state to either of the local product states $|++\rangle$, $|--\rangle$, $|+-\rangle$ or $| - + \rangle$, the succeeding measurement of S_{xx} is no longer identical to that of the original input state. In other words, the local projective measurement of σ_z on either Alice's or Bob's qubit is not compatible with S_{xx} .

Another strategy for the measurement of spin products is the *nonlocal* measurement making use of additional entanglement shared by Alice and Bob. Suppose Alice and Bob share a maximally entangled Bell state (ebit):

$$|\xi\rangle_M = |\Phi^+\rangle_M. \quad (14)$$

The suffix M indicates that it is used as a meter to measure the system state. Alice and Bob use each qubit in $|\xi\rangle_M$ as a meter (probe) to measure their system qubit. To do so, each of them makes a CNOT gate between her/his qubits, as shown in Fig. 2, and then makes a projective σ_z measurement on her/his meter qubit. Note that when her/his initial meter qubit was fixed as $|+\rangle$, the measurement would be the projective local measurement of σ_z . After the CNOT gates, the initial state $|\psi\rangle_S \otimes |\Phi^+\rangle_M$ is converted to

$$|\psi\rangle_S \otimes |\Phi^+\rangle_M \rightarrow (c_1|\Phi^+\rangle + c_2|\Phi^-\rangle)_S \otimes |\Phi^+\rangle_M + (c_3|\Psi^+\rangle + c_4|\Psi^-\rangle)_S \otimes |\Psi^+\rangle_M. \quad (15)$$

Let Alice's (Bob's) outcome be z_A (z_B). The first term of the right hand side of (15) corresponds to the case where the measurement outcome is $(z_A z_B) = (++)$ or $(--)$, while the second term to $(+-)$ or $(-+)$. By simply taking a product of the local meter outcomes of Alice and Bob, $z_A z_B = m = \pm 1$ is obtained and thus the measurement of S_{zz} is complete. The measurement operators M_\pm for $m = \pm 1$ are

$$M_+ = \Pi(\Phi^+) + \Pi(\Phi^-), \quad (16)$$

$$M_- = \Pi(\Psi^+) + \Pi(\Psi^-). \quad (17)$$

The corresponding POVMs are identical to that presented in (12) and (13). M_\pm project the system state to the eigenspaces of $m = \pm 1$ presented in (6) and (7),

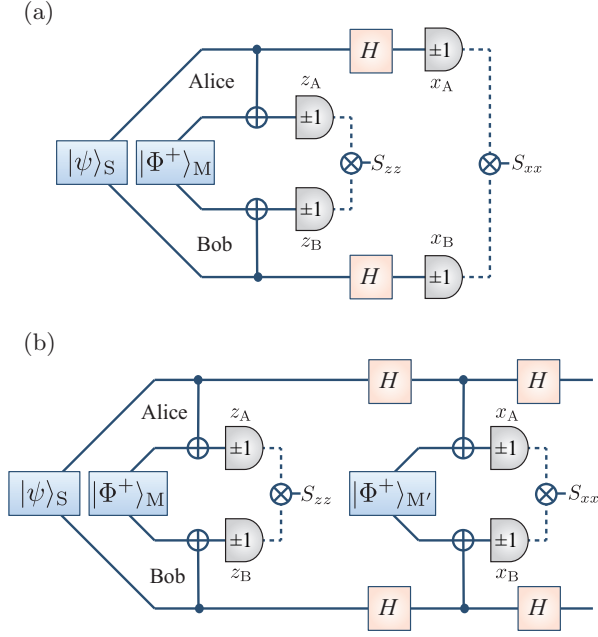


FIG. 3: Two schemes for the complete Bell measurement making use of sequential spin product measurements, S_{zz} and S_{xx} . (a) With nonlocal S_{zz} and local S_{xx} measurements. (b) With nonlocal measurements of S_{zz} and S_{xx} . Note that the scheme (b) acts as a complete Bell filter.

respectively. Note that the system state is still a superposition of $|\Phi^+\rangle$ and $|\Phi^-\rangle$ (or $|\Psi^+\rangle$ and $|\Psi^-\rangle$), preserving sufficient information for the succeeding S_{xx} measurement.

COMPLETE BELL MEASUREMENT AND A BELL FILTER

Subsequent to the nonlocal measurement of S_{zz} described above, Alice and Bob make a S_{xx} measurement on the system state. As shown in (9) and (10), the measurement of S_{xx} combined with S_{zz} discriminates $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi^+\rangle$, or $|\Psi^-\rangle$ from (15). In this way, a complete Bell measurement can be carried out.

For the S_{xx} measurement, either local or nonlocal strategy can be used. If only the measurement outcome matters, the simple local strategy shown in Fig. 3(a) is appropriate. In this case, the POVMs E_{mn} , where suffixes m and n indicate the measurement outcomes of the preceding S_{zz} and the following S_{xx} , respectively, are written as

$$E_{++} = \Pi(\Phi^+), \quad E_{+-} = \Pi(\Phi^-), \quad (18)$$

$$E_{-+} = \Pi(\Psi^+), \quad E_{--} = \Pi(\Psi^-). \quad (19)$$

On the other hand, if the system state at the output should be preserved in one of the resultant eigenstates given in (9) and (10), i.e., one of the Bell bases, Alice and

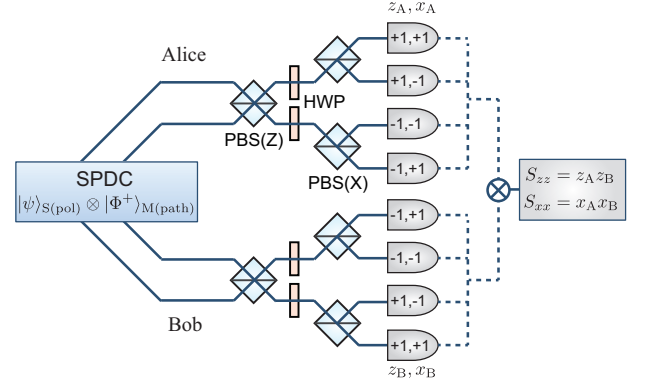


FIG. 4: Proposed optical implementation of the complete Bell measurement scheme shown in Fig. 3(a). Photon pairs are generated by spontaneous parametric down-conversion (SPDC) so that their path qubits are entangled in $|\Phi^+\rangle_M$. A polarization beamsplitter (PBS) works as a CNOT gate between the polarization (system) and the path (meter) qubits. The outcomes of σ_z (z_A and z_B on Alice's and Bob's qubits, respectively) are encoded in the output paths of PBS(Z). After passing through the half-wave plate (HWP), which acts as a Hadamard gate, the outcomes of σ_x (x_A and x_B) are encoded in the output paths of PBS(X).

Bob can use nonlocal strategy at a cost of an additional ebit, as shown in Fig. 3(b). In this case, the measurement operators M_{mn} are found to be

$$M_{++} = \Pi(\Phi^+), \quad M_{+-} = \Pi(\Phi^-), \quad (20)$$

$$M_{-+} = \Pi(\Psi^+), \quad M_{--} = \Pi(\Psi^-). \quad (21)$$

Again, the corresponding POVMs are identical to that presented in (18) and (19). Thus, the system state is projected into one of the Bell basis depending on the measurement outcomes. This procedure functions as a complete *Bell filter*, where the output state will be either one of the Bell bases indicated by the measurement outcome.

It is noteworthy that, in both strategies, all of the outcomes are deterministically obtained and thus the Bell measurement proposed here is complete and deterministic, at the cost of requiring one (for the Bell measurement) or two (for the Bell filter) ebit(s) as a resource.

PROPOSED EXPERIMENTS

The measurement schemes described above are applicable to any physical qubits between which we can prepare entanglement and a CNOT operation. However, in cases where we can directly make a nonlocal CNOT operation between qubits held in Alice and Bob, we could employ a simpler scheme as shown, for instance, in Fig. 1 to implement the Bell measurement. Nevertheless, our scheme is still useful when we are not able to use nonlocal CNOT, or when we need the function of the Bell filter

as well as the Bell measurement.

Another situation where our schemes may be useful is the case of photonic qubits. It is known that with linear optics we cannot implement deterministic CNOT gates between individual photonic qubits [7]. As a result, to date, we could not implement deterministic Bell measurement with linear optics. Nevertheless, employing the scheme described in this paper we will be able to implement the deterministic and complete Bell measurement between photonic qubits.

Suppose we provide a pair of photons to Alice and Bob, as shown in Fig. 4. The photons' polarizations constitute the system state $|\psi\rangle_S$ of interest. In order to measure the nonlocal spin products on their polarizations, we prepare their path degrees of freedom, i.e., path qubits, in the maximally entangled Bell state $|\Phi^+\rangle_M$. Entanglement in the path degrees of freedom could be directly generated by spatial entanglement between photons generated by parametric down-conversion [9], or could be converted from time-bin entanglement [10]. When the polarization qubits are also entangled, it is called a hyperentangled state [11]. Between the polarization qubit and the path qubit, Alice and Bob employ CNOT gates using polarizing beamsplitters (PBS). Thus, the nonlocal measurement of S_{zz} on the photons' polarization qubits is implemented and the measurement outcomes are encoded in photons' output paths. Then Alice and Bob carry out the local σ_x measurement for their polarization qubits using, for instance, two additional PBSs. This part implements the local measurement strategy of S_{xx} . At the last stage, Alice detects her photon at one of her four output paths, as does Bob at one of his four output paths. From the path information, they know the result of S_{zz} and S_{xx} , and thus the complete Bell measurement is carried out in a deterministic way. One drawback of this linear optics implementation is that we use an ebit implemented in the path degree of freedom of the photon pair. As a result, it is difficult to apply this method to Bell measurement between independent photons as in a case of quantum teleportation. Nonetheless, this method will be useful in many situations of quantum technologies where nonlocal Bell measurement is an essential resource.

CONCLUSIONS

In this letter we see that complete and deterministic Bell measurement is possible in terms of nonlocal spin product measurements. Although the Bell measurement requires an ebit as a resource, the scheme is readily realizable using present technologies. In particular, we propose an optical implementation using linear optics. In addition, a complete Bell filter, which requires a couple of ebits and thus seems a bit more difficult to implement, is also possible. These schemes will be useful in quantum technologies where nonlocal Bell measurement is indis-

pensable.

Furthermore, the measurement protocol of nonlocal spin products can be extended to measurements at weak and any intermediate measurement strength [8]. Thus, it would be possible to realize generalized measurements of nonlocal spin products and Bell measurement at any measurement strength. In this context, strength-variable measurements of photon polarization and the measurement uncertainty relations have been demonstrated [12–17]. By extending the protocols described here, it would be possible to explore measurement uncertainty relations in the nonlocal product observables.

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